Journal of Statistical Physics, Vol. 94, Nos. 3/4, 1999

Simple Random Walks on Tori

Ya. G. Sinai[1]

Received March 26, 1998

We consider a Markov chain whose phase space is a d-dimensional torus. A point x jumps to $x+\omega$ with probability $p(x)$ and to $x-\omega$ with probability $1-p(x)$. For Diophantine ω and smooth p we prove that this Markov chain has an absolutely continuous invariant measure and the distribution of any point after n steps converges to this measure.

KEY WORDS: Markov chain; homological equation; Levy excursion; stable law.

1. INTRODUCTION

Consider Markov Chain on the d-dimensional torus Tor^d where a moving point jumps from $x \in \mathrm{Tor}^d$ to $x \pm \omega$ with probabilities $p(x)$ and $(1-p(x))$, respectively. Here $\omega = (\omega_1, \omega_2, ..., \omega_d) \in \mathrm{Tor}^d$ is fixed. We shall call such Markov chains simple random walks on tori. In the one-dimensional case a point wanders along the unit circle jumping from x to $x \pm \omega$.

We shall impose the following conditions:

1° The point ω is Diophantine, i.e. for some positive K, γ

$$\inf_{m \in \mathbb{Z}^1} |(\omega, n) - m| \geqslant \frac{K}{|n|^\gamma}$$

where $n = (n_1, ..., n_d) \in \mathbb{Z}^d$, $|n| = \sum_{i=1}^d |n_i| \neq 0$. The shift on Tor^d by ω is denoted by T, i.e. $Tx = x + \omega$ (the addition mod 1).

2° Let $H^r(\mathrm{Tor}^d) = H^r$ be the space of continuous functions on Tor^d such that for their Fourier expansions $f = \sum_{s \in \mathbb{Z}^d} f_s e^{2\pi i (s, x)}$ the series $\sum_{s \in \mathbb{Z}^d} |f_s| \cdot |s|^r < \infty$. Then $0 < p(x) < 1$, $x \in \mathrm{Tor}^d$ and $p \in H^r$ for some $r > 2\gamma$.

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[1] Department of Mathematics, Princeton University, Princeton, New Jersey, and Landau Institute of Theoretical Physics, Moscow, Russia.

0022-4715/99/0200-0695$16.00/0

We shall prove for this Markov chain that under formulated above conditions it has a unique invariant measure which is absolutely continuous with respect to the Lebesgue measure. The density of this measure π satisfies the equation

$$\pi(x) = p(T^{-1}x)\,\pi(T^{-1}x) + (1 - p(Tx))\,\pi(Tx) \tag{1}$$

Let us show that (1) always has a solution.

Definition. Simple random walk is symmetric if

$$\int_{\mathrm{Tor}^d} \ln p(x)\,dx = \int_{\mathrm{Tor}^d} \ln(1 - p(x))\,dx.$$

Otherwise it is called non-symmetric.

Consider first symmetric case. Here the homological equation

$$\frac{p(x)}{1 - p(x)} = \frac{h(x)}{h(T^{-1}x)} \tag{2}$$

has a positive solution $h \in H^{r-\gamma}$. Let us check that the function $\pi_0(x) = h(x)/p(x)$ satisfies (1). We have

$$\begin{aligned} &p(T^{-1}x)\,\pi_0(T^{-1}x) + (1 - p(Tx))\,\pi_0(Tx) \\ &\quad = h(T^{-1}x) + \frac{1 - p(Tx)}{p(Tx)} \cdot h(Tx) = h(T^{-1}x) + h(x) \end{aligned}$$

But $h(T^{-1}x) + h(x) = h(x)/p(x)$. Indeed, $h(x)(1/p(x) - 1) = h(T^{-1}x)$ or

$$\frac{1 - p(x)}{p(x)} = \frac{h(T^{-1}x)}{h(x)}$$

which is our homological equation (2).

In the non-symmetric case we write

$$\frac{p(x)}{1 - p(x)} = \lambda\,\frac{h(x)}{h(T^{-1}x)} \tag{3}$$

where $\ln \lambda = \int \ln p(x)\,dx - \int \ln(1 - p(x))\,dx$ and $h \in H^{r-\gamma}$. Assume that $\lambda > 1$. Consider the equation

$$\lambda^{-1} r^{(+)}(Tx) - r^{(+)}(x) = \frac{1}{h(x)} \tag{4}$$

For $\lambda \neq 1$ it always has a solution since no small denominators arise. We shall see later that $r^{+}(x) > 0$. Let us show that $\pi_0(x) = r^{(+)}(x)\, h(x)/p(x)$ satisfies (1). We have

$$\frac{r^{(+)}(x)\, h(x)}{p(x)} = r^{(+)}(T^{-1}x)\, h(T^{-1}x) + \frac{(1-p(Tx))\, r^{+}(Tx)\, h(Tx)}{p(Tx)}$$

Using (3) we can write

$$\frac{r^{(+)}(x)}{p(x)} = r^{(+)}(T^{-1}x) \cdot \frac{h(T^{-1}x)}{h(x)} + \lambda^{-1} r^{+}(Tx)$$

The right-hand side can be rewritten using (4) as

$$\begin{aligned}
&\frac{r^{(+)}(T^{-1}x)\, h(T^{-1}x)}{h(x)} + \lambda^{-1} r^{+}(Tx) \\
&= \left(\lambda^{-1} r^{(+)}(x) - \frac{1}{h(T^{-1}x)} \right) \frac{h(T^{-1}x)}{h(x)} + \lambda^{-1} r^{(+)}(Tx) \\
&= \lambda^{-1} r^{(+)}(x)\, \frac{h(T^{-1}x)}{h(x)} - \frac{1}{h(x)} + \left(\frac{1}{h(x)} + r^{(+)}(x) \right) \\
&= r^{(+)}(x) \left(\lambda^{-1} \frac{h(T^{-1}x)}{h(x)} + 1 \right) \\
&= r^{(+)}(x) \left(\frac{1-p(x)}{p(x)} + 1 \right) \\
&= \frac{r^{(+)}(x)}{p(x)}
\end{aligned}$$

The case $\lambda < 1$ is considered in the same way by replacing T by T^{-1}. The probabilistic meaning of the functions h, $r^{(+)}$ will become clear later.

The main result of this paper is the following theorem.

Theorem 1. Under the conditions formulated above the invariant measure of simple random walk on Tor^d is unique. Even more, if $P_k(T^m x)$ is the probability that after k steps the moving point is at $T^m x$, $|m| \leqslant k$, then for $k \to \infty$ the probability measures $P_k = \{P_k(T^m x)\}$ converge weakly to $\pi_x(x)\, dx$.

This theorem is proven in Sections 2, 3, and 4 for the symmetric case. In Section 5 we explain the needed changes in the non-symmetric case. In Sections 6 we formulate some extensions of Theorem 1 and mention several open problems.

2. RECURRENCE OF SYMMETRIC RANDOM WALKS

Fix $x \in \mathrm{Tor}^d$ and consider simple random walk $b = \{b(k), k \geqslant 0\}$ on $\mathbb{Z}^1$ where $b(0) = 0$ and the probabilities to go from m to $m \pm 1$ are $p(T^m x)$, $1 - p(T^m x)$, respectively. We shall show in this section that in the symmetric case random walk b is recurrent.

Take any segment $[k_1, k_2], k_1 < k_2$ and introduce the probabilities $R_k^{(\pm)}$ of random walks b which go out of k and reach k_2, respectively k_1, earlier than k_1, respectively k_2. These probabilities satisfy the following set of recurrent relations and boundary conditions

$$\begin{aligned} &R_k^{(+)} = p(T^\kappa x)\, R_{k+1}^{(+)} + (1 - p(T^\kappa x))\, R_{k-1}^{(+)}, \qquad k_1 < k < k_2, \\ &R_{k_1}^{(+)} = 0, \qquad R_{k_2}^{(+)} = 1 \end{aligned} \tag{5}$$

$$\begin{aligned} &R_k^{(-)} = p(T^\kappa x)\, R_{k+1}^{(-)} + (1 - p(T^\kappa x))\, R_{k-1}^{(-)}, \qquad k_1 < k < k_2, \\ &R_{k_1}^{(-)} = 1, \qquad R_{k_2}^{(-)} = 0 \end{aligned} \tag{6}$$

To construct needed solutions we have to find two linearly independent solutions. One is trivial: $R \equiv 1$. Let us check that the other one is given by the formulas

$$R_{k_1} = 0,\ R_k = \sum_{i = k_1 + 1}^{k} h(T^i x), \qquad k > k_1$$

We must check that for $k > k_1$

$$R_k = p(T^k x)\, R_{k+1} + (1 - p(T^\kappa x)\, R_{k-1}$$

which in our case is reduced to

$$0 = p(T^k x)\, h(T^{k+1} x) - (1 - p(T^k x))\, h(T^k x)$$

But this is equivalent to (2). Now it is easy to see that

$$R_k^{(+)} = \frac{\sum_{i = k_1 + 1}^{k} h(T^i x)}{\sum_{k_1 < i \leqslant k_2} h(T^i x)}, \qquad k > k_1$$

and

$$R_k^{(-)} = \frac{\sum_{i = k}^{k_2 - 1} h(T^i x)}{\sum_{k_1 \leqslant i < k_2} h(T^i x)}$$

are needed solutions of (2), (3). Take $k = 0, k_2 = 1$. We immediately see that $\lim_{k_1 \to -\infty} R_0^{(+)} = 1$. In the same way for $k_1 = -1$ the limit

$\lim_{k_2 \to \infty} R_0^{(-)} = 1$. This shows that random walk b with probability 1 returns to the origin and infinitely many times, i.e. random walk b is recurrent.

3. LEVY EXCURSIONS AND THEIR DISTRIBUTIONS

Consider the probabilities $p_{2n}^{(+)}(x)$ of random walks b such that $b(0) = 0$, $b(k) > 0$ for $1 \leqslant k < 2n$, $b(2n) = 0$. In view of recurrence $\sum_{n \geqslant 1} p_{2n}^{(+)}(x) = p(x)$. We shall use generating functions $\varphi^{(+)}(\theta, x) = \sum_{n \geqslant 1} \theta^{2n} p_{2n}^{(+)}(x)$, $|\theta| \leqslant 1$. We have

$$p_2^{(+)}(x) = p(x)(1 - p(Tx)),$$
$$p_{2n}^{(+)}(x) = p(x) \sum_{s \geqslant 1} \sum_{n_1 + n_2 + \cdots n_s = n-1} p_{2n_1}^{(+)}(Tx)\, p_{2n_2}^{(+)}(Tx) \cdot \cdots$$
$$\cdot\, p_{2n_s}^{(+)}(Tx)(1 - p(Tx)), \qquad n > 1$$

Multiplying both sides by θ^{2n} and adding over n we arrive at the equation (see also [S2])

$$\varphi^{(+)}(\theta; x) = p(x)(1 - p(Tx))\, \theta^2 \left(1 + \sum_{k \geqslant 1} (\varphi^{(+)}(\theta; Tx))^k\right)$$
$$= \frac{p(x)(1 - p(Tx))\, \theta^2}{1 - \varphi^{(+)}(\theta; Tx)} \tag{7}$$

Put $\theta_1 = \sqrt{1 - \theta^2}$, $\psi^{(+)}(\theta_1; x) = h(x)(1 - \varphi^{(+)}(\theta; x)/p(x))$. We have from (7) the following equation for $\psi^{(+)}(\theta_1; x)$

$$\psi^{(+)}(\theta_1; x) = \frac{\theta_1^2(x) + \psi^{(+)}(\theta_1; Tx)}{1 + \dfrac{\psi^{(+)}(\theta_1; Tx)}{h(x)}}$$

which we rewrite in the form

$$\psi^{(+)}(\theta_1; x) = h(x) - (1 - \theta_1^2)\, h(x) \left(1 - \frac{\psi^{(+)}(\theta_1; Tx)}{h(x)} \cdot \frac{1}{1 + \dfrac{\psi^{(+)}(\theta_1; Tx)}{h(x)}}\right)$$
$$= \theta_1^2 h(x) + (1 - \theta_1^2) \frac{1}{\dfrac{1}{h(x)} + \dfrac{1}{\psi^{(+)}(\theta_1; Tx)}} \tag{8}$$

Introduce the one-dimensional map $Q_x(z)=\theta_1^2 h(x)+(1-\theta_1^2)\,1/(1/h(x)+1/z)$, $z\geqslant 0$. The dependence on θ_1 is not mentioned explicitly. From (8) for any $k>0$

$$\psi^{(+)}(\theta_1;x)=Q_x(\psi^{(+)}(\theta_1;Tx))=Q_x\circ Q_{Tx}\circ\cdots\circ Q_{T^\kappa x}(\psi^{(+)}(\theta_1;T^{\kappa+1}x))$$

The maps Q_x have the following properties:

1° Any Q_x is a contraction:

$$|Q_x(z')-Q_x(z'')|=(1-\theta_1^2)\left[\frac{1}{\dfrac{1}{h(x)}+\dfrac{1}{z'}}-\frac{1}{\dfrac{1}{h(x)}+\dfrac{1}{z''}}\right]$$

$$=(1-\theta_1^2)\,\frac{z''-z'}{\left(1+\dfrac{z'}{h(x)}\right)\left(1+\dfrac{z''}{h(x)}\right)}$$

and the needed statement follows from positivity of z and h.

The property implies, in particular, the uniqueness of solutions of (8) and (7).

2° There exists a positive constant K_1 such that for all sufficiently small θ_1

$$0\leqslant Q_x(z)\leqslant K_1\theta_1 \qquad \text{if} \quad 0\leqslant z\leqslant K_1\theta_1$$

Indeed, put $H=\max_x h(x)$, $H_1=\min h(x)$. We have

$$Q_x(z)=\theta_1^2 h(x)+(1-\theta_1^2)\,\frac{z}{1+\dfrac{z}{h(x)}}$$

$$=\theta_1^2 h(x)+(1-\theta_1^2)\left(z-\frac{\dfrac{z^2}{h(x)}}{1+\dfrac{z}{h(x)}}\right)$$

$$\leqslant\theta_1^2 H+K_1\theta_1-\frac{(1-\theta_1^2)\,K_1^2\theta_1^2}{H_1+K_1\theta_1}\leqslant K_1\theta_1$$

provided that K_1 is large enough.

Using 1° and 2° we can write $\psi^{(+)}(\theta_1; x)$ as an infinite fraction

$$\psi^{(+)}(\theta_1; x) = \theta_1^2(x) + (1-\theta_1^2)\cfrac{1}{\cfrac{1}{h(x)} + \cfrac{1}{\theta_1^2 h(Tx) + (1-\theta_1^2)\cfrac{1}{\cfrac{1}{h(Tx)}} + \cdots}}$$

which should be understood as the limit of finite fractions. It can be rewritten as a continued fraction consisting of the infinite number of fragments

$$(1-\theta_1^2)^k\, \theta_1^2 h(T^\kappa x) + \cfrac{1}{\cfrac{1}{(1-\theta_1^2)^{\kappa+1}\, h(T^\kappa x)}}$$

3° If $h \in H^{r-\gamma}$ then $\psi^{(+)}(\theta_1; x) \in C^{r-2\gamma}(\mathrm{Tor}^d)$ as a function of x.

Indeed, if $h \in H^{r-\gamma}$ then $h \in C^{r-\gamma}(\mathrm{Tor}^d)$ and the statement follows by direct differentiation of the infinite fraction giving $\psi^{(+)}(\theta_1; x)$.

4° Let ω be Diophantine. Then $\psi^{(+)}(\theta_1; x)$ has the following representation

$$\psi^{(+)}(\theta_1; x) = a^{(+)}\theta_1 + \theta_1^2 f^{(+)}(\theta_1; x)$$

where $a^{(+)}$ is a constant, $a^{(+)} \leqslant K_1$ (see 2°), $f^{(+)} \in C^{r-\gamma}(\mathrm{Tor}^d)$.

Proof. We can write $\psi^{(+)}(\theta_1; x) = \theta_1 a^{(+)} + \theta_1 F^{(+)}(\theta_1; x)$ where

$$\theta_1 a^{(+)} = \int_{\mathrm{Tor}^d} \psi^{(+)}(\theta_1; x)\, dx \leqslant \theta_1 K_1$$

in view of 2°, $|F(\theta_1; x)| \leqslant 2K_1$. Rewrite (8) as follows

$$\begin{aligned} F^{(+)}(\theta_1; x) &- F^{(+)}(\theta_1; Tx) \\ &= \theta_1 h(x) - \theta_1^2 a^{(+)} - \theta_1^2 F^{(+)}(\theta_1; Tx) \\ &= \frac{\dfrac{\theta_1(1-\theta_1^2)(a^{(+)} + F^{(+)}(\theta_1; Tx))}{h(x)}}{1 + \dfrac{\theta_1 a^{(+)} + \theta_1 F^{(+)}(\theta_1; Tx)}{h(x)}} = \theta_1 G^{(+)}(\theta_1; x) \end{aligned} \tag{9}$$

where $G^{(+)}(\theta_1; x) \in C^{r-\gamma}(\mathrm{Tor}^d)$. Since ω is Diophantine the solution of (9) can be written as $F^{(+)}(\theta_1; x) = \theta_1 f^{(+)}(\theta_1; x)$, $f^{(+)}(\theta_1; x) \in C^{r-\gamma}(\mathrm{Tor}^d)$.

Return back to $\psi^{(+)}(\theta; x)$. Our previous analysis gives the following representation for $\psi^{(+)}$:

$$\varphi^{(+)}(\theta; x) = p(x)\left(1 - \frac{\psi^{(+)}(\sqrt{1-\theta^2}; x)}{h(x)}\right)$$
$$= p(x)\left(1 - \frac{a^{(+)}\sqrt{1-\theta^2}}{h(x)} + (1-\theta^2)\, f^{(+)}\left(\sqrt{1-\theta^2}; x\right)\right) \quad (10)$$

The Tauberian theorem for generating functions (see [F]) implies

$$p_{2n}^{(+)}(x) = \frac{p(x)}{\text{const}}\frac{1}{n^{3/2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

In other words, the probabilities $p_{2n}^{(+)}(x)$ decay in the same way as the similar probabilities for the usual symmetric simple random walk.

The asymptotics of probabilities $p_{2n}^{(-)}(x)$ of such b that $b(1) = -1$, $b(k) < 0$ for $1 \leqslant k \leqslant 2n$, $b(2n) = 0$ is investigated in the same way. For the corresponding generating function $\varphi^{(-)}(\theta; x)$ we can write

$$\varphi^{(-)}(\theta; x) = (1 - p(x))\left(1 - \frac{a^{(-)}\sqrt{1-\theta^2}}{h(x)} + (1-\theta^2)\, f^{(-)}(\sqrt{1-\theta^2}; x)\right) \quad (11)$$

For the generating function of the moment of the first return to the origin $\varphi(\theta; x) = \varphi^{(+)}(\theta; x) + \varphi^{(-)}(\theta; x)$ we have

$$\varphi(\theta; x) = 1 - \frac{a\sqrt{1-\theta^2}}{h(x)} + (1-\theta^2)\, f(\sqrt{1-\theta^2}; x) \quad (12)$$

where $a = a^{(+)}p(x) + a^{(-)}(1-p(x))$, $f(\sqrt{1-\theta^2}; x) = f^{(+)}(\sqrt{1-\theta^2}; x)\, p(x) + f^{(-)}(\sqrt{1-\theta^2}; x)(1-p(x))$.

4. PROOF OF THEOREM 1 IN THE SYMMETRIC CASE

Return back to our random walk on Tor^d. After k steps the moving point can be in any point $T^m x$, $|m| \leqslant k$ and the probability to be at $T^m x$ is the probability $P_k(m)$ that $b(k) = m$. We shall study the asymptotics of $P_\kappa(m)$ as $k \to \infty$, $m \to \infty$ so that m^2/k tends to a constant $z \neq 0$. Assume for definiteness that $z > 0$. Introduce the following random moments $\mathcal{T}_{i,k}(b) = \max\{n \mid b(n) = i, n \leqslant k\}$, $i = 0, 1, ..., m$. It is clear that $b(\mathcal{T}_{0,k}(b)) = 0$,

$b(n) > 0$ for $n > \mathcal{T}_{0,k}(b)$. The difference $\mathcal{T}_{i+1,k}(b) - \mathcal{T}_{i,k}(b) = \tau_{i+1}$ consists of several positive excursions starting at i and of the last step from i to $i+1$. Let $v_i \geqslant 0$ be the number of these excursions, $\xi_{i1}, ..., \xi_{iv_1}$ are their lengths. If $v_i = 0$ then the particle jumps from i to $i+1$ and after that does not return to i before k. We can write (see [S2])

$$P_k(m) = \sum_{v_0, v_1, ..., v_m} \sum_{\sum_{i=1}^{m} (2k_1^{(i)} + \cdots + 2k_{v_i}^{(i)}) = k - m} p(x) \prod_{s=1}^{v_0} p_{2k_s^{(0)}}(x) \cdot \prod_{i=1}^{m-1} \left[p(T^i x) \cdot \prod_{i=1}^{v_i} p^{(+)}_{2k_i^{(i)}}(T^i x) \right] \tag{13}$$

Introduce the generating function $\phi(\theta; m) = \sum P_k(m)\, \theta^k$, $|\theta| < 1$. We have from (13)

$$\begin{aligned} \phi(\theta; m) &= (p(T^m x)\, \theta)^{-1} \frac{p(x)\, \theta}{1 - \varphi(\theta; x)} \cdot \prod_{i=1}^{m} (p(T^i x)\, \theta) \\ &\quad \cdot \sum_{v_1, ..., v_m,\, v_i \geqslant 0} \prod_{i=1}^{m} \varphi^{(+)}(\theta; T^i x)^{v_i} \\ &= (p(T^m x)\, \theta)^{-1}\, \theta^m \frac{\theta p(x)}{1 - \varphi(\theta; x)} \prod_{i=1}^{m} \frac{(p(T^i x))}{1 - p(T^i x)} \\ &\quad \cdot \prod_{i=1}^{m} \frac{(1 - p(T^i x))}{1 - \varphi^{(+)}(\theta; t^i x)} \end{aligned} \tag{14}$$

Now we use again the symmetry of our random walk (see Section 1), which yields

$$\prod_{i=1}^{m} \frac{p(T^i x)}{1 - p(T^i x)} = \frac{h(T^m x)}{h(x)}$$

Each of the functions $1 - p(T^i x)/1 - \varphi^{(+)}(\theta; t^i x)$ is the generating function of the distribution of $\sum_{j=1}^{v_i} \xi_{ij} = \xi_i$ belonging to the domain of attraction of the stable one-sided law with exponent $\alpha = \frac{1}{2}$ (see [GK] and [F]). This follows easily from the results in Section 3. Therefore the distribution of the normed sum $1/m^2 \sum_{i=1}^{m} \zeta_i = \zeta$ converges to this law. One can see this also from the expression for the characteristics function of ζ equal to

$$
\begin{aligned}
f(t) &= Ee^{it\zeta} = Ee^{i\,t/m^2\cdot\sum_{i=1}^m \zeta_i}\\
&= \prod_{i=1}^m \frac{1-p(T^i x)}{1-\varphi^{(+)}(e^i\, t/m^2;\, T^i x)}\\
&= \prod_{i=1}^m \frac{1}{1+\dfrac{p(T^i x)\, a^{(+)}\sqrt{2}\sqrt{it}\,(1+O(1/m^2))}{(1-p(T^i x))\, h(T^i x)\cdot m}}\\
&\qquad \times \exp\left\{ it\,\frac{1}{m}\sum_{i=1}^m \frac{p(T^i x)\, a^{(+)}\sqrt{2}}{(1-p(T^i x))\, h(T^i x)}\cdot(1+o(1))\right\}
\end{aligned}
$$

The average $1/m\sum_{i=1}^m p(T^i x)\, a^{(+)}\sqrt{2}/(1-p(T^i x))\, h(T^i x) = a^{(+)}\sqrt{2}1/m$ $\sum_{i=1}^m 1/h(T^{i-1}x)$ in view of (2) converges for every x and $m\to\infty$ to $\sigma_0 = a^{(+)}\sqrt{2}\int dx/h(x)$. Thus $f(t)$ converges to $\exp\{\sqrt{it\sigma_0}\}$ which is the characteristic function of the above-mentioned law.

In (14) we have also the factor $1/1-\varphi(\theta;x) = \sum_{n=0}^\infty \varphi^n(\theta;x)$. Our arguments below imply easily the "arcsin"-law (see [F]) for random walk b but we shall not discuss this in detail. Each $\varphi^n(\theta;x)$ is the generating function for the sum of n independent identically distributed random variables whose distribution belongs to the domain of attraction of the same stable law. Again putting $\exp\{i\,t/m^2\}$ we can write using (12)

$$
\begin{aligned}
&\varphi^n(e^{i\,t/m^2};\, x)\prod_{i=1}^m \frac{1-p(T^i x)}{1-\varphi^{(+)}(e^{i\,t/m^2};\, T^i x)}\\
&\qquad = \left(1-\frac{a\sqrt{it}}{h(x)\,m}\right)^n\cdot e^{\sqrt{it}\sigma_0}\cdot(1+o(1))\\
&\qquad = e^{-an\sqrt{it}/h(x)\,m}\cdot e^{\sqrt{it}\sigma_0}(1+o(1))
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n\geqslant 0}\varphi^n(e^{i\,t/m^2};\, x)\prod_{i=1}^m \frac{1-p(T^i x)}{1-\varphi^{(+)}(e^{i\,t/m^2};\, T^i x)}\\
&\qquad = \sum_{n\geqslant 0} e^{-an\sqrt{it}/h(x)\,m}\prod_{i=1}^m \frac{1-p(T^i x)}{1-\varphi^{(+)}(e^{it/m^2};\, T^i x)}(1+o(1))\\
&\qquad = \frac{mh(x)}{a\sqrt{it}}\, e^{\sqrt{it}\sigma_0}(1+o(1)) \qquad (15)
\end{aligned}
$$

Now we can use the local limit theorem of probability theory (see [GK]) which says that

$$P\left\{\frac{1}{m^2}\sum_{i=1}^{m}\zeta_i = z\right\} = \frac{g(\sigma_1^{-1}z)\,\sigma_1^{-1}}{m^2}\,(1+\varepsilon(m,z))$$

where g is the standard Gaussian density and $\varepsilon(m,z)$ tends to zero as $m\to\infty$ uniformly for all z in any fixed interval $[z_1, z_2]$, $0<z_1<z_2<\infty$. It is easy to check that (15) implies

$$P_\kappa(m) = \frac{h(T^m x)}{p(T^m x)}\cdot e^{-m^2/2k\sigma_1}\cdot\frac{1}{\sqrt{2\pi\sigma_1}}\,(1+o(1))$$

for some $\sigma_1>0$ where $o(1)$ is uniformly small in any finite interval of values of $z=m/\sqrt{k}$. This implies Theorem 1 in the symmetric case.

5. NON-SYMMETRIC RANDOM WALKS

In the non-symmetric case we begin with the definition of the mean drift. We use the same systems of equations (5) and (6) and again want to find two linearly independent solutions. As before, one is $R_k\equiv 1$. We shall try to find another one in the form $R_k=\mu^\kappa r(T^k x)$ for some unknown μ and r. We have for them the equation

$$\mu r(Tx)-r(x)=\mu^{-1}\,\frac{1-p(x)}{p(x)}\,(\mu r(x)-r(T^{-1}x)) \tag{16}$$

Choose $\mu=\lambda^{-1}$ (see (3)) so that

$$\mu\,\frac{1-p(x)}{p(x)}=\frac{h(T^{-1}x)}{h(x)}$$

If $p\in H^r$ and ω is Diophantine, the function $h\in H^{r-\gamma}$ and (16) leads to

$$\lambda^{-1}r(Tx)-r(x)=\frac{1}{h(x)} \tag{17}$$

The equation (17) always has a solution $h\in H^{r-\gamma}$ since no small denominators arise. We can write explicit expressions for $R_k^{(\pm)}$:

$$R_k^{(+)}=\frac{\lambda^{-k}r(T^k x)-\lambda^{-k_1}r(T^{k_1}x)}{\lambda^{-k_2}r(T^{k_2}x)-\lambda^{-k_1}r(T^{k_1}r(T^{k_1}x)},\qquad k\geqslant k_1 \tag{18$'$}$$

$$R_k^{(-)}=\frac{\lambda^{-k_2}r(T^{k_2}x)-\lambda^{-k}r(T^k x)}{\lambda^{-k_2}r(T^{k_2}x)-\lambda^{-k_1}r(T^{k_1}r(T^{k_1}x)},\qquad k\leqslant k_2 \tag{18$''$}$$

Let $\lambda < 1$. Take $k = 0$, $k_2 = 1$, $k_1 \to -\infty$. From (18′), (18″) it follows that $R^{(+)} \to \lambda^{-1} r(x)/r(Tx) < 1$. This means that we have a mean drift to the left. If $\lambda > 1$ a mean drift is to the right.

Consider for definiteness the case $\lambda > 1$. It is easy to see that with probability 1 the limit $\lim_{k \to \infty} b(k) = \infty$. A positive excursion of length $2n$ is a part of b such that $b(m) = 0$, $b(2n+m) = 0$, $b(k) > 0$ for $m < k < 2n + m$. The probabilities $p_{2n}^{(+)}(x)$ of the positive excursions of the length $2n$ satisfy the same equations as in the symmetric case (see Section 3) and for the corresponding generating function $\varphi^{(+)}(\theta; x)$ we have the same equation (7). However in this case $\varphi^{(+)}(1, x) = \sum_{n=1}^{\infty} p_{2n}^{(+)}(x) = p(x)\, q(x)$ where $0 < q(x) = \lambda^{-1} r(Tx)/r(x)$ is the conditional probability that random walk b goes out of 1 and eventually comes back to 0. It follows from the last inequality that $r(x) > 0$.

We shall not switch from θ to θ_1 as we did in Section 3 and shall consider $\psi^{(+)}(\theta; x) = h(x)(1 - \varphi^{(+)}(\theta; x)/p(x))$. It satisfies the equation

$$\psi^{(+)}(\theta; x) = \frac{(1-\theta^2)\, h(x) + \psi^{(+)}(\theta; Tx)\, \lambda}{1 + \dfrac{\psi^{(+)}(\theta; Tx)\, \lambda}{h(x)}}$$

or

$$\psi^{(+)}(\theta; x) = (1-\theta^2)\, h(x) + \cfrac{1}{\cfrac{1}{h(x)} + \cfrac{1}{\psi^{(+)}(\theta; Tx)\, \lambda}} \tag{19}$$

From this equation it follows that

$$\psi^{(+)}(1; x) = \cfrac{1}{\cfrac{1}{h(x)} + \cfrac{1}{\lambda h(Tx)} + \cfrac{1}{\lambda^2 h(T^2 x)} + \cdots}$$

and $0 < \psi^{(+)}(1; x) < \infty$ since $\lambda > 1$.

The analysis of this equation is quite similar to the symmetric case. We write $\psi^{(+)}(\theta; x) = \psi^{(+)}(1; x)(1 + \delta(\theta; Tx))$ and for $\delta(\theta; x)$ we have from (19)

$$\delta(\theta; x) = \frac{1}{1 + \dfrac{\psi_0(Tx)\, \lambda}{h(x)}}\, \delta(\theta; Tx) + \cdots$$

where dots mean terms of higher order of smallness. This shows that in a small neighborhood of $\psi^{(+)}(1, x)$ the mapping $Q_z(\theta; z) = (1-\theta^2)\, h(x) + \theta^2/1/h(x) + 1/z\lambda$ is a contraction and in some neighborhood $U = (1-\varepsilon, 1+\varepsilon)$ the solution $\psi^{(+)}(\theta; x)$, $\theta \in U$ of (19) is a real analytic function on U. It implies in particular that the probabilities $p_{2n}(x)$ decay in this case exponentially.

Again we study the asymptotic behavior of probabilities $P_\kappa(m)$. We can write the same expression (13) and (14) takes the form

$$\psi(\theta; m) = p(T^m x) \cdot \theta)^{-1} \cdot \theta^m \prod_{i=1}^{m} \frac{1-\varphi^{(+)}(1; T^i x)}{1-\varphi^{(+)}(\theta; T^i x)}$$
$$\cdot \prod_{i=1}^{m} \cdot \frac{p(T^i x)}{1-\varphi^{(+)}(1, T^i x)} \cdot \frac{\theta p(x)}{1-\varphi(\theta; x)}$$

We already saw (see above) that $\varphi^{(+)}(1, x) = p(x)\, \lambda^{-1} r(Tx)/r(x)$ and

$$\begin{aligned}
1-\varphi^{(+)}(1, x) &= 1 - \frac{p(x)\, \lambda^{-1} r(Tx)}{r(x)} \\
&= 1 - \frac{p(x)(r(x) + h^{-1}(x))}{r(x)} \\
&= (1-p(x)) \left(1 - \frac{p(x)}{1-p(x)} \cdot \frac{1}{h(x)\, r(x)}\right) \\
&= (1-p(x)) \cdot \left(1 - \frac{\lambda}{(r(x)\, h(T^{-1}x)}\right) \\
&= (1-p(x)) \left(\lambda^{-1} r(x) - \frac{1}{h(T^{-1}x)}\right) \cdot \frac{\lambda}{r(x)} \\
&= (1-p(x))\, r(T^{-1}x) \cdot \frac{\lambda}{r(x)}
\end{aligned}$$

Thus

$$\begin{aligned}
\prod_{i=1}^{m} \frac{p(T^i x)}{1-\varphi^{(+)}(1, x)} &= \prod_{i=1}^{m} \frac{p(T^i x)\, r(T^i x)}{(1-p(T^i x))\, r(T^{i-1}x)^{-1}} \\
&= \prod_{k=1}^{m} \frac{h(T^i x)\, r(T^i x)}{h(T^{i-1}x)\, r(T^{i-1}x)} = \frac{h(T^m x)\, r(T^m x)}{h(x)\, r(x)}
\end{aligned}$$

The ratio $1-\varphi^{(+)}(1, T^i x)/1-\varphi^{(+)}(\theta; T^i x)$ is a generating function of a positive random variable with an exponentially decaying distribution. So the product $\theta^m \prod_{i=1}^m 1-\varphi^{(+)}(1; T^i x)/1-\varphi^{(+)}(\theta; T^i x)$ has the usual Gaussian asymptotics and we can write

$$P_\kappa(m) = \frac{h(T^m x)\, r(T^m x)}{p(T^m x)}\, e^{-1/2\,(m-ak)^2/\sigma} C(x)(1+\varepsilon(m,k)) \tag{20}$$

Here a is the mean drift, $\sigma > 0$ is a constant, $C(x)$ is a number depending on $x, \varepsilon(m, k)$ tend to zero as $k \to \infty$ and m remains in $\mathcal{O}(\sqrt{k})$-neighborhood of $a \cdot k$. It is clear that (20) implies Theorem 1.

6. SOME GENERALIZATIONS AND OPEN PROBLEMS

The same methods give the existence and uniqueness of stationary measures for one-dimensional diffusion processes with smooth local characteristics taking place on orbits of Diophantine groups of shifts on Tor^d.

The main problem considered in this paper is a particular case of the following more general problem. Suppose that we have a measure-preserving automorphism T acting on a measure space $(M, \mathscr{M}, \mu)$ and $p < 1$ is positive a.e.. Consider Markov chain where a point $x \in M$ jumps to Tx with probability $p(x)$ and to $T^{-1}x$ with probability $1 - p(x)$. Problem: does this Markov chain have an invariant measure equivalent to μ. We believe that in the case of T with strong mixing properties like Anosov transitive diffeomorphisms the answer is negative. Probably this case is connected with random walks in random environments (see [S1]). It would be interesting to extend the results of this paper to groups $\mathbb{Z}^k$, $k < d$ acting on Tor^d and to Markov chains where a point can jump from x to $T^i x$, $|i| \leqslant i_0$.

ACKNOWLEDGMENTS

The financial support from NSF, Grant No. DMS-9706794 and RFFI, Grant No. 96-01-0037 are acknowledged. I thank V. Kaloshin and W. Schlag for useful remarks.

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Communicated by J. L. Lebowitz